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2004 J. Phys. A: Math. Gen. 37 5625

(http://iopscience.iop.org/0305-4470/37/21/008)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 5625–5634 PII: S0305-4470(04)74631-4

Laplacian eigenmodes for the three-sphere

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Received 14 January 2004
Published 12 May 2004
Online at stacks.iop.org/JPhysA/37/5625
DOI: 10.1088/0305-4470/37/21/008

Abstract

The vector space $\mathcal{V}^k$ of the eigenfunctions of the Laplacian on the three-sphere S^3, corresponding to the same eigenvalue $\lambda_k = -k\,(k+2)$, has dimension $(k+1)^2$. After recalling the standard bases for $\mathcal{V}^k$, we introduce a new basis B3, constructed from the reductions to S^3 of a peculiar homogeneous harmonic polynomial involving null vectors. We give the transformation laws between this basis and the usual hyper-spherical harmonics. Thanks to the quaternionic representations of S^3 and $SO(4)$, we are able to write explicitly the transformation properties of B3, and thus of any eigenmode, under an arbitrary rotation of $SO(4)$. This offers the possibility of selecting those functions of $\mathcal{V}^k$ which remain invariant under a chosen rotation of $SO(4)$. When the rotation is a holonomy transformation of a spherical space S^3/Γ, this gives a method for calculating the eigenmodes of S^3/Γ, which remains an open problem in general. We illustrate our method by (re-)deriving the eigenmodes of lens and prism space. In a companion paper, we present the derivation for dodecahedral space.

PACS numbers: 02.30.Gp, 02.40.−k

1. Introduction

The eigenvalues of the Laplacian Δ of S^3 are of the form $\lambda_k = -k\,(k+2)$, where $k \in \mathbb{N}$. For a given value of k, they span the eigenspace $\mathcal{V}^k$ of dimension $(k+1)^2$. This vector space constitutes the $(k+1)^2$-dimensional irreducible representation of $SO(4)$, the isometry group of S^3.

There are two commonly used bases (hereafter B1 and B2) for $\mathcal{V}^k$ which generalize in some sense (see below) the usual spherical harmonics $Y_{\ell m}$ for the two-sphere. The functions of these bases have a friendly behaviour under some of the rotations of $SO(4)$; this generalizes the property of the $Y_{\ell m}$ to be eigenfunctions of the angular momentum operator in $\mathbb{R}^3$. However, these functions show no special properties under the *general* rotation of $SO(4)$.

Except for some cases (lens and prism spaces, see below), the search for the eigenmodes of the spherical spaces of the form S^3/Γ remains an open problem. The dimension of the eigenspaces has been however calculated in [4] (this provides a cross-check of the validity of our results). Since those are eigenmodes of S^3 which remain invariant under the rotations of Γ, it is clear that this search requires an understanding of the rotation properties of the basis functions under $SO(4)$.

The task in this paper is to examine the rotation properties of the eigenfunctions of $\mathcal{V}^k$, as a preparatory work for the search for eigenfunctions of S^3/Γ (in particular for dodecahedral space). This will be done through the introduction of a new basis B3 of $\mathcal{V}^k$ (in the case k even), for which the rotation properties can be explicitly calculated: following a new procedure (that was already applied to S^2 in [6]) we generate a system of coherent states on $\mathcal{V}^k$. We extract from it a basis B3 of $\mathcal{V}^k$, which seems to have been ignored in the literature, and present original properties. Each function Φ^k_{IJ} of this basis B3 is defined as (the reduction to S^3 of) a homogeneous harmonic polynomial in $\mathbb{R}^4$, which takes the very simple form $(X \cdot N)^k$. Here, the dot product extends the Euclidean (scalar) dot product of $\mathbb{R}^4$ to its complexification $\mathbb{C}^4$, and N is a null vector of $\mathbb{C}^4$, that we specify below. After defining these functions, we show that they form a basis of $\mathcal{V}^k$, and we give the explicit transformation formulae between B2 and B3.

The properties of the basis B3 differ from those of the two other bases, and make it more convenient for particular applications. In particular, it is possible to calculate explicitly its rotation properties, under an arbitrary rotation of $SO(4)$, by using their quaternionic representation (section 3). This allows us to find those functions which remain invariant under an arbitrary rotation. In section 4, we apply these results to (re-)derive the eigenmodes of lens and prism spaces.

2. Harmonic functions

A function f on S^3 is an eigenmode (of the Laplacian) if it satisfies $\Delta f = \lambda f$. It is known that eigenvalues are of the form $\lambda_k = -k(k+2)$, $k \in \mathbb{N}$. The corresponding eigenfunctions generate the eigen(vector)space $\mathcal{V}^k$, of dimension $(k+1)^2$, which realizes an irreducible unitary representation of the group $SO(4)$.

First basis. I call B1 the most widely used basis for $\mathcal{V}^k$ provided by the hyperspherical harmonics

$$B1 \equiv (\mathcal{Y}_{k\ell m}) \qquad \ell = 1, \ldots, k \qquad m = -\ell, \ldots, \ell. \tag{1}$$

It generalizes the usual spherical harmonics $Y_{\ell m}$ on the sphere (note that $\mathcal{Y}_{k\ell m} \propto Y_{\ell m}$). In fact, it can be shown ([1, 2, p 240, 3]) that a basis of this type exists on any sphere S^n. Moreover, [2, 3] show that the B1 basis for S^n is 'naturally generated' by the B1 basis for S^{n-1}. In this sense, the B1 basis for S^3 is generated by the usual spherical harmonics $Y_{\ell m}$ on the two-sphere S^2.

The generation process involves harmonic polynomials constructed from null complex vectors (see below). The basis B1 is in fact based on the reduction of the representation of $SO(4)$ to representations of $SO(3)$: each $\mathcal{Y}_{k\ell m}$ is an eigenfunction of an $SO(3)$ subgroup of $SO(4)$ which leaves a selected point of S^3 invariant. This makes these functions useful when one considers the action of that particular $SO(3)$ subgroup. But they show no simple behaviour under a general rotation. We will no longer use this basis.

Second basis. By group theoretical arguments, a different (ON) basis of $\mathcal{V}^k$ is constructed in [1], which is specific to S^3:

$$B2 \equiv \left(T_{k;m_1,m_2}\right) \qquad m_1, m_2 = -k/2, \ldots, k/2 \tag{2}$$

where m_1 and m_2 vary independently by integers (and, thus, take integral or semi-integral values according to the parity of k). In the spirit of the construction referred to above, B2 may be seen as generated from a different choice of spherical harmonics on S^2. The bases B1 and B2 appear respectively adapted to the systems of hyperspherical and toroidal (see below) coordinates to describe S^3.

Formula (27) of [1], reduced to the three-sphere, shows that the elements of this basis take a very convenient form if we use *toroidal coordinates* (as they are called in [8]) on the three-sphere S^3: (χ, θ, ϕ) spanning the range $0 \leqslant \chi \leqslant \pi/2$, $0 \leqslant \theta \leqslant 2\pi$ and $0 \leqslant \phi \leqslant 2\pi$. They are conveniently defined (see [8] for a more complete description) from an isometric embedding of S^3 in $\mathbb{R}^4$ (as the hypersurface $x \in \mathbb{R}^4$; $|x| = 1$):

$$\begin{cases} x^0 = r\cos\chi\cos\theta \\ x^1 = r\sin\chi\cos\phi \\ x^2 = r\sin\chi\sin\phi \\ x^3 = r\cos\chi\sin\theta \end{cases}$$

where (x^μ), $\mu = 0, 1, 2, 3$, is a point of $\mathbb{R}^4$. As shown in [8], these coordinates appear naturally associated with some isometries.

Very simple manipulations show that, with these coordinates, each eigenfunction of B2 takes the form

$$T_{k;m_1,m_2}(X) \equiv t_{k;m_1,m_2}(\chi)\,\mathrm{e}^{\mathrm{i}\ell\theta}\,\mathrm{e}^{\mathrm{i}m\phi} \qquad (3)$$

where the $t_{k;m_1,m_2}(\chi)$ are polynomials in $\cos\chi$ and $\sin\chi$ and we wrote, for simplification, $\ell \equiv m_1 + m_2$, $m \equiv m_2 - m_1$.

To have a convenient expression, we substitute this formula into the harmonic equation expressed in coordinates χ, θ, ϕ. This leads to a second-order differential equation (cf equation (15) of [8]). The solution is proportional to a Jacobi polynomial: $t_{k;m_1,m_2}(\chi) \propto \cos^\ell\chi\,\sin^m\chi\,P_d^{m,\ell}(\cos 2\chi)$, $d \equiv k/2 - m_2$. Thus, we have the final expression for the basis B2

$$T_{k;m_1,m_2}(X) = C_{k;m_1,m_2}[\cos\chi\,\mathrm{e}^{\mathrm{i}\theta}]^\ell[\sin\chi\,\mathrm{e}^{\mathrm{i}\phi}]^m P_d^{(m,\ell)}[\cos(2\chi)] \qquad (4)$$

with $C_{k;m_1,m_2} \equiv \frac{\sqrt{(k+1)}}{\pi}\sqrt{\frac{(k/2+m_2)!(k/2-m_2)!}{(k/2+m_1)!(k/2-m_1)!}}$ from normalization requirements (the variation ranges of m_1 and m_2 imply that the quantities under the factorial sign are integral and positive). Note also the useful proportionality relations:

$$\cos^\ell\chi\,\sin^m\chi\,P^{(m,\ell)}_{\frac{k-\ell-m}{2}}(\cos 2\chi) \propto \cos^\ell\chi\,\sin^{-m}\chi\,P^{(-m,\ell)}_{\frac{k-\ell+m}{2}}(\cos 2\chi)$$
$$\propto \cos^{-\ell}\chi\,\sin^m\chi\,P^{(m,-\ell)}_{\frac{k+\ell-m}{2}}(\cos 2\chi).$$

The term $\zeta^m\xi^n \equiv \mathrm{e}^{\mathrm{i}\ell\theta}\,\mathrm{e}^{\mathrm{i}m\phi}$ in (4) defines the rotation properties of $T_{k;m_1,m_2}$ under a specific subgroup of $SO(4)$. These properties generalize the properties of the spherical harmonics on the two-sphere S^2, to be eigenfunctions of the rotation operator P_x. This advantage has been used in [8] to calculate (from a slightly different basis) the eigenmodes of lens or prism spaces (see section 4). However, the $T_{k;m_1,m_2}$ have no simple rotation properties under the general rotation of $SO(4)$. This motivates the search for a different basis of $\mathcal{V}^k$.

Note that the basis functions $T_{k;m_1,m_2}$ have also been introduced in [2] (p 253), with their expression in Jacobi polynomials. Note also that they are the complex counterparts of those proposed in [8] (their equation (19)) to find the eigenmodes of lens and prism spaces. The variation range of the indices m_1, m_2 here (equation (2)) is equivalent to their condition

$$|\ell| + |m| \leqslant k \qquad \ell + m = k \qquad \mathrm{mod}\,(2) \qquad (5)$$

through the correspondence $\ell = m_1 + m_2$, $m = m_2 - m_1$.

2.1. Complex null vectors

A complex vector $Z \equiv (Z^0, Z^1, Z^2, Z^3)$ is an element of $\mathbb{C}^4$. We extend the Euclidean scalar product in $\mathbb{R}^4$ to the complex (non-Hermitian) inner product $Z \cdot Z' \equiv \sum_\mu Z^\mu (Z')^\mu$, $\mu = 0, 1, 2, 3$. A *null* vector N is defined as having a zero norm $N \cdot N \equiv \sum_\mu N^\mu N^\mu = 0$ (in which case, it may be considered as a point on the isotropic cone in $\mathbb{C}^4$). It is well known that polynomials of the form $(X \cdot N)^k$, homogeneous of degree k, are harmonic if and only if N is a null vector. This results from

$$\Delta_0 (X \cdot N)^k \equiv \sum_\mu \partial_\mu \partial_\mu (X \cdot N)^k = k(k-1) \left(\sum_\mu (N^\mu N^\mu) \right) (X \cdot N)^{k-2} = 0$$

where Δ_0 is the Laplacian of $\mathbb{R}^4$. Thus, the restrictions of such polynomials belong to $\mathcal{V}^k$. As we mentioned above, particular null vectors have been used in [2, 3] to generate the bases B1 and B2.

To construct a third basis B3, let us first define a family of null vectors

$$N(a, b) \equiv (\cos a, \mathrm{i} \sin b, \mathrm{i} \cos b, \sin a) \tag{6}$$

indexed by two angles a and b describing the unit circle (they define coherent states in $\mathbb{R}^4$).

The polynomial $[X \cdot N(a, b)]^k$ is harmonic and, thus, can be decomposed with respect to the basis B2. It is easy to check that, like the scalar product $X \cdot N(a, b)$, this polynomial depends on a and b only through the combinations $\mathrm{e}^{\mathrm{i}(\theta - a)}$ and $\mathrm{e}^{\mathrm{i}(\phi + b)}$, with their conjugates. This implies that its decomposition with respect to B2 takes the form

$$[X \cdot N(a, b)]^k = \sum_{m_1, m_2} P_{k;m_1,m_2} T_{k;m_1,m_2}(X)\, \mathrm{e}^{-\mathrm{i}a(m_1+m_2)}\, \mathrm{e}^{\mathrm{i}b(m_2-m_1)} \tag{7}$$

where the coefficients $P_{k;m_1,m_2}$ do not depend on a, b. Now we intend to find a basis of $\mathcal{V}^k$ in the form of such polynomials.

2.2. A new basis

2.2.1. Roots of unity. To do so, we consider the $(k+1)$th complex roots of unity which are the powers ρ^I of

$$\rho \equiv \mathrm{e}^{\frac{2\mathrm{i}\pi}{k+1}} \equiv \cos\alpha + \mathrm{i}\sin\alpha \qquad \alpha \equiv \frac{2\pi}{k+1}. \tag{8}$$

We recall the fundamental property, which will be widely used thereafter:

$$\sum_{n=0}^{k} \rho^{nI} = \begin{cases} k+1 & \text{if } (k+1) \mid I \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

where $(k+1) \mid I$ means I is an integral multiple of $k+1$.

In a given frame, we consider the family of null vectors

$$N_{IJ} \equiv N(I\alpha, J\alpha) = (\cos I\alpha, \mathrm{i}\sin J\alpha, \mathrm{i}\cos J\alpha, \sin I\alpha) \qquad I, J = 0, \ldots, k \tag{10}$$

and we define the functions Φ^k_{IJ}: $\Phi^k_{IJ}(X) \equiv (X \cdot N_{IJ})^k$. We substitute such a function into equation (7) to obtain its expansion in the basis B2. Then we multiply both terms by $\rho^{I(m_1+m_2)-J(m_2-m_1)}$. Making the summations over I, J (each varying from 0 to k), and using (9), we obtain, in the case where k is even, that we assume hereafter (when k is odd, the Φ^k_{IJ}

are not linearly independent, and thus do not form a basis):

$$\mathcal{T}_{k;m_1,m_2} = \frac{1}{(k+1)^2} \sum_{I,J=0}^{k} \rho^{I(m_1+m_2)-J(m_2-m_1)} \Phi^k_{IJ} \tag{11}$$

where $\mathcal{T}_{k;m_1,m_2} \equiv P_{k;m_1,m_2} T_{k;m_1,m_2}$.

This gives the decomposition of any $T_{k;m_1,m_2}$ (and thus, of any harmonic function) as a sum of the $(k+1)^2$ polynomials Φ^k_{IJ}, providing the new basis of $\mathcal{V}^k$:

$$B3 \equiv \left(\Phi^k_{IJ}\right) \qquad I, J = 0, \ldots, k \quad (k \text{ even}). \tag{12}$$

The coefficients $P_{k;m_1,m_2}$ involved in the transformation are calculated in the appendix. We obtain easily the reciprocal formula expressing the change of basis:

$$\Phi^k_{IJ} = \sum_{m_1,m_2=-k/2}^{k/2} \mathcal{T}_{k;m_1,m_2}\, \rho^{-I(m_1+m_2)+J(m_2-m_1)}. \tag{13}$$

3. Rotations in $\mathbb{R}^4$

3.1. Matrix representations

The isometries of S^3 are the rotations in the embedding space $\mathbb{R}^4$. In the usual matrix representation, a rotation is represented by a 4×4 orthogonal matrix $g \in SO(4)$, acting on the 4-vector (x^μ) by matrix product.

In the complex matrix representation, a point (vector) of $\mathbb{R}^3$ is represented by the 2×2 complex matrix

$$X \equiv \begin{bmatrix} W & \mathrm{i}Z \\ \mathrm{i}\bar{Z} & \bar{W} \end{bmatrix} \qquad W \equiv x^0 + \mathrm{i}x^3 \qquad Z \equiv x^1 + \mathrm{i}x^2 \in \mathbb{C}.$$

A rotation g is represented by two complex 2×2 matrices (G_L, G_R), so that its action takes the form $X \mapsto G_L X G_R$ (matrix product). The two matrices G_L and G_R belong to $SU(2)$. Since $SU(2)$ identifies with S^3, any matrix G_L or G_R is of the same form as the matrix X above. Since $SU(2)$ is also the set of unit norm quaternions, there is a quaternionic representation for the action of $SO(4)$.

3.2. Quaternionic notation

Let us note j_μ, $\mu = 0, 1, 2, 3$, the basis of real quaternions (the j_μ correspond to the usual $1, i, j, k$ but we do not use this notation here). We have $j_0 = 1$. A general quaternion is $q = q^\mu j_\mu = q^0 + q^i j_i$ (with summation convention; the index i takes the values 1, 2, 3; the index μ takes the values 0, 1, 2, 3), and $q^\mu \in \mathbb{R}$. Its quaternionic conjugate is $\bar{q} \equiv q^0 - q^i j_i$. The scalar product is $q_1 \cdot q_2 \equiv (q_1\bar{q}_2 + q_2\bar{q}_1)/2$, giving the quaternionic norm $|q|^2 = \frac{q\bar{q}}{2} = \sum_\mu (q^\mu)^2$.

We represent any point $x = (x^\mu)$ of $\mathbb{R}^4$ by the quaternion $q_x \equiv x^\mu j_\mu$. The points of the (unit) sphere S^3 correspond to unit quaternions, $|q|^2 = 1$. Hereafter, all quaternions will be unitary (if not otherwise indicated). It is easy to see that, using the coordinates above, a point of S^3 is represented by the quaternion $\cos\chi\,\dot{\zeta} + \sin\chi\,\dot{\xi}\, j_1$, where we define dotted quantities, like $\dot{\zeta} \equiv \cos\theta + j_3 \sin\theta$, $\dot{\xi} \equiv \cos\phi + j_3 \sin\phi$, as the quaternionic analogues of the complex numbers $\zeta = \cos\theta + \mathrm{i}\sin\theta$ and $\xi = \cos\phi + \mathrm{i}\sin\phi$, i.e., with the imaginary i replaced by the quaternion j_3.

In quaternionic notation, the rotation $g : x \mapsto gx$ is represented by a pair of unit quaternions (Q_L, Q_R) such that $q_x \mapsto q_{gx} = Q_L q_x Q_R$.

Complex quaternions, null quaternions. The null vectors N introduced above do not belong to $\mathbb{R}^4$ but to $\mathbb{C}^4$. Thus, they cannot be represented by real quaternions, but by complex quaternions. They are defined exactly like the usual quaternions, but with complex rather than real coefficients. Note that the pure imaginary i does not coincide with any of the j_μ, but commutes with all of them. Also, complex conjugation (star) and quaternionic conjugation (bar) must be carefully distinguished. Then it is easy to see that the (null) vectors N_{IJ} defined above correspond to the complex quaternions $n_{IJ} \equiv \dot{\rho}^I + i j_2 \dot{\rho}^J$. Note that $|n_{IJ}|^2 = 0$.

In quaternionic notation, the basis functions are expressed as

$$\Phi_{IJ}(x) = (N_{IJ} \cdot x)^k = \langle n_{IJ} \cdot q_X \rangle^k = \left(\frac{n_{IJ} \bar{q_X} + q_X \bar{n}_{IJ}}{2} \right)^k. \tag{14}$$

Quaternionic notation will help us to check how the basis functions are transformed by the rotations of $SO(4)$.

3.3. Rotations of functions

With any rotation g, is associated its action $\mathbf{R}_g$ on functions: $\mathbf{R}_g : f \mapsto \mathbf{R}_g f; \mathbf{R}_g f(x) \equiv f(gx)$. Let us apply this action to the basis functions:

$$\mathcal{R}_g \Phi_{IJ}(x) = \Phi_{IJ}(gx) = \langle n_{IJ} \cdot (Q_L q_x Q_R) \rangle^k. \tag{15}$$

We consider a function on S^3 also as a function on the set of unit quaternions (q_x is the unit quaternion associated with the point x of S^3). On the other hand, we may expand this function with respect to the basis:

$$\mathbf{R}_g \Phi_{IJ} \equiv \sum_{ij=0}^{k} G_{IJ}^{ij}(g) \Phi_{ij}. \tag{16}$$

The coefficients $G_{IJ}^{ij}(g)$ of the expansion, that we intend to calculate, completely encode the action of the rotation g on the basis B3, and thus on $\mathcal{V}^k$.

To proceed, we introduce three auxiliary complex quaternions:

$$\alpha \equiv 1 + i j_3 \qquad \beta \equiv j_1 - i j_2 = (1 - i j_3) j_1 \qquad \text{and} \qquad \delta \equiv -j_1 - i j_2.$$

They have zero norm and obey the properties $\langle \alpha \cdot n_{IJ} \rangle = \rho^I$, $\langle \bar{\alpha} \cdot n_{IJ} \rangle = \rho^{-I}$, $\langle \beta \cdot n_{IJ} \rangle = \rho^J$, $\langle \delta \cdot n_{IJ} \rangle = \rho^{-J}$. Let us now compute relation (16) for the specific quaternion $\alpha + R\bar{\alpha} + S\beta + T\delta$, with R, S, T arbitrary real numbers:

$$(\mathcal{A} + R\mathcal{A}' + S\mathcal{B} + T\mathcal{D})^k = \sum_{ij} G_{IJ}^{ij}(g) < (\rho^i + R\rho^{-i} + S\rho^j + T\rho^{-j})^k \tag{17}$$

where $\mathcal{A} \equiv \langle Q_L \alpha Q_R \cdot n_{IJ} \rangle$, $\mathcal{A}' \equiv \langle Q_L \bar{\alpha} Q_R \cdot n_{IJ} \rangle$, $\mathcal{B} \equiv \langle Q_L \beta Q_R \cdot n_{IJ} \rangle$, $\mathcal{D} \equiv \langle Q_L \delta Q_R \cdot n_{IJ} \rangle$ characterize the rotation. (Note that these quantities depend on I and J.)

We expand and identify the powers of the variables R, S, T:

$$\mathcal{A}^q \mathcal{A}'^{p-q} \mathcal{B}^r \mathcal{D}^{k-p-r} = \sum_{ij} G_{IJ}^{ij}(g) \rho^{i(2q-p)} \rho^{j(2r-k+p)}.$$

This holds for $0 \leqslant q \leqslant p$, $0 \leqslant r \leqslant k-p$, $0 \leqslant p \leqslant k$. After the definition of the new indices $A \equiv q + r$, $B \equiv q - r + k - p$, which both vary from 0 to k, the previous equation takes the form

$$\left(\frac{\mathcal{A}\mathcal{B}}{\mathcal{A}'\mathcal{D}} \right)^{A/2} \left(\frac{\mathcal{A}\mathcal{D}}{\mathcal{A}'\mathcal{B}} \right)^{B/2} \left(\frac{\mathcal{A}\mathcal{A}'}{\mathcal{B}\mathcal{D}} \right)^{p/2} \left(\frac{\mathcal{A}'\mathcal{B}\mathcal{D}}{\mathcal{A}} \right)^{k/2} = \sum_{ij} G_{IJ}^{ij}(g) \rho^{i(A+B-k)+j(A-B)}.$$

This holds for any value of A, B, p. A consequence is that $\mathcal{A}\mathcal{A}' = \mathcal{B}\mathcal{D}$, which can be checked directly. Finally,

$$\mathcal{U}^A\mathcal{V}^B\left(\mathcal{A}'\right)^k = \sum_{ij} G_{IJ}^{ij}(g)\rho^{i(A+B-k)}\rho^{j(A-B)}$$

with $\mathcal{U} \equiv \left(\frac{\mathcal{B}}{\mathcal{A}'}\right)$, $\mathcal{V} \equiv \left(\frac{\mathcal{A}}{\mathcal{B}}\right)$.

Taking into account the properties of the roots of unity, this equation has the solution

$$G_{IJ}^{ij} = \frac{\left(\mathcal{A}'\right)^k}{(k+1)^2}\sum_{A,B=0}^{k}\rho^{-i(A+B-k)}\rho^{-j(A-B)}\mathcal{U}^A\mathcal{V}^B. \qquad (18)$$

When a rotation is specified, there is no difficulty in computing the associated values of $\mathcal{A}', \mathcal{U}, \mathcal{V}$, and thus of these coefficients which completely encode the transformation properties of the basis functions of $\mathcal{V}^k$ under $SO(4)$.

In the next section, we apply these results to rederive the eigenmodes of lens or prism space. In the companion paper [7], we take for g the generators of Γ, the group of holonomies of the dodecahedral space. This allows the selection of the invariant functions, which constitute its eigenmodes.

4. Lens and prism spaces

The eigenmodes for lens and prism spaces have been found in [8]. Here we derive them again for the illustration of our method.

4.1. Lens space

The lens space $L(p,q) = S^3/\Gamma$, where Γ is a cyclic group of order q (see, for instance, [5]). It is defined by a unique generator, whose form is given below in terms of the prime integers p and q, which are also relatively prime. A holonomy transformation of a lens space takes the form, in complex notation,

$$G_L = \begin{bmatrix} e^{i\frac{\psi_1+\psi_2}{2}} & 0 \\ 0 & e^{-i\frac{\psi_1+\psi_2}{2}} \end{bmatrix} \qquad G_R = \begin{bmatrix} e^{i\frac{\psi_1-\psi_2}{2}} & 0 \\ 0 & e^{-i\frac{\psi_1-\psi_2}{2}} \end{bmatrix}. \qquad (19)$$

Its action on a vector of $\mathbb{R}^4$ takes the form

$$X \equiv \begin{bmatrix} W & iZ \\ i\bar{Z} & \bar{W} \end{bmatrix} \mapsto G_L X G_R = \begin{bmatrix} W\,e^{i\psi_1} & iZ\,e^{i\psi_2} \\ i\bar{Z}\,e^{-i\psi_2} & \bar{W}\,e^{-i\psi_1} \end{bmatrix}. \qquad (20)$$

In this simple case, $W \equiv x^0 + ix^3$, $Z \equiv x^1 + ix^2$ are transformed into $W\,e^{i\psi_1}$ and $Z\,e^{i\psi_2}$, respectively. This corresponds to the quaternionic notation

$$Q_L = \dot{w}_1\dot{w}_2 \qquad Q_R = \dot{w}_1/\dot{w}_2 \qquad \dot{w}_i \equiv \cos(\psi_i/2) + j_3\sin(\psi_i/2). \qquad (21)$$

The rotation is expressed in the simplest way in the toroidal coordinates, since it acts as $\theta \mapsto \theta + \psi_1, \phi \mapsto \phi + \psi_2$. From expression (4) of the basis functions (B2), it gives their transformation law:

$$\mathbf{R}_g : T_{k;m_1,m_2} \mapsto T_{k;m_1,m_2}\,e^{\ell\psi_1+m\psi_2}$$

where $\ell = m_1+m_2$ and $m = m_1-m_2$ as in (5). This leads directly to the invariance condition $\ell\psi_1 + m\psi_2 = 0 \bmod 2\pi$. Using the standard notation for a lens space $L(p,q)$, namely

$$\psi_1 = 2\pi/p \qquad \psi_2 = 2\pi q/p$$

we are led to the conclusion:

> the eigenmodes of lens space $L(p, q)$ are all linear combinations of $T_{k;\underline{m_1},\underline{m_2}}$, where the underlining means that the indices satisfy the condition $\underline{m_1} + \underline{m_2} + q(\underline{m_2} - \underline{m_1}) = 0$, modulo$(p)$.

4.2. Prism space

The prism space is the quotient S^3/D_P^*, where D_P^* is the binary dihedral group of order $4P$ (see for instance [5], which uses the notation D_{4P}^* instead of D_P^*, or [8]). The two generators are single action rotations ($G_R = 0$). The first generator, analogous to the lens case above, with $\psi_1 = \psi_2 = 2\pi/2P$, provides the first condition $\ell + m = 0$, mod $2P$ which takes the form

$$\underline{m_2} = 0 \qquad \text{mod } P. \tag{22}$$

Since m_2 varies from $-k/2$ to $k/2$ by integers (2), this implies that k must be even.

The second generator has the complex matrix form $G = G_L = \left[\begin{smallmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{smallmatrix}\right]$, which corresponds to the quaternion $Q_L = Q = -j_1$. Easy calculations lead to $\mathcal{A} = -\rho^J$, $\mathcal{A}' = \rho^{-J}$, $\mathcal{B} = \rho^I$, $\mathcal{D} = -\rho^{-I}$. Substituting these into (18) gives

$$G_{IJ}^{ij} = \frac{\rho^{(i-J)k}}{(k+1)^2} \sum_{A,B=0}^{k} \rho^{A(-i-j+I+J)+B(-i+j-I+J)} (-1)^B. \tag{23}$$

This formula, together with those expressing the change of basis between B2 and B3, allows us to return to the rotation properties of the basis B2 which take the simple form:

$$\mathbf{R} : \mathcal{T}_{k;m_1,m_2} \mapsto (-1)^{m_2+k/2} \mathcal{T}_{k;m_1,-m_2}. \tag{24}$$

It results immediately that the G-invariant functions are combinations of $\mathcal{T}_{k;m_1,m_2} + (-1)^{m_2+k/2} \mathcal{T}_{k;m_1,-m_2}$.

Finally,

> the eigenfunctions of the prism space are combinations of $\mathcal{T}_{k;m_1,\underline{m_2}} + (-1)^{\underline{m_2}+k/2}$ $\mathcal{T}_{k;m_1,-\underline{m_2}}$, $\forall m_1$; k even, where the underlining means that $\underline{m_2}$ satisfies the condition $\underline{m_2} = 0$, modulo(P).

According to the parity of $k/2$, the functions $\mathcal{T}_{k;m_1,0}$ are included or not, from which simple counting gives the multiplicity as $(k+1)(1 + [k/2P])$, for k even ($[\cdots]$ means integral value), $(k+1)[k/2P]$, for k odd, in accordance with [4].

5. Conclusion

We have shown that $\mathcal{V}^k$, the space of eigenfunctions of the Laplacian of S^3 with a given eigenvalue λ_k (k even) admits a new basis B3. In contrast to standard bases (B1 and B2) which show specific rotation properties under selected subgroups of $SO(4)$, it is possible to calculate explicitly the rotation properties of B3 under any rotation of the group $SO(4)$, as well as to calculate the functions invariant under this rotation. This opens the door to the calculation of eigenmodes of spherical space. The eigenfunctions of lens and prism spaces had been calculated in [8], by using a basis related to B2 (its real, rather than complex, version). We rederived them to illustrate the properties of the bases.

In a companion paper [7], we applied these results to the search for the eigenfunctions of the dodecahedral space S^3/Γ, where $\Gamma = D_P^*$ is the binary dihedral group of order $4P$. These functions, formerly unknown, are the eigenfunctions of S^3 which remain invariant under the elements of Γ.

Appendix

Let us evaluate the function

$$Z^k_{\ell m}(X) \equiv \sum_{IJ=0}^{k} \rho^{\ell I - Jm} \Phi^k_{IJ}(X)$$
$$= 2^{-k} \sum_{IJ} \rho^{\ell I - mJ} \left[\cos\chi \left(\zeta\rho^{-I} + \frac{1}{\zeta\rho^{-I}}\right) + \sin\chi \left(\xi\rho^{J} - \frac{1}{\xi\rho^{J}}\right)\right]^k \tag{A.1}$$

where we defined $\zeta \equiv e^{i\theta}$ and $\xi \equiv e^{i\phi}$. After the expansion of the power with the binomial coefficients, the sum becomes

$$\sum_{IJ} \rho^{\ell I - mJ} \sum_{p=0}^{k} \binom{k}{p} \left[\cos\chi \left(\zeta\rho^{-I} + \frac{1}{\zeta\rho^{-I}}\right)\right]^{k-p} \left[\sin\chi \left(\xi\rho^{J} - \frac{1}{\xi\rho^{J}}\right)\right]^{p}. \tag{A.2}$$

Let us write the identities

$$\rho^{\ell I} \left(\zeta\rho^{-I} + \frac{1}{\zeta\rho^{-I}}\right)^{k-p} = \sum_{r=0}^{k-p} \binom{k-p}{r} \zeta^{2r+p-k} \rho^{-I(2r+p-k-\ell)} \tag{A.3}$$

$$\rho^{-mJ} \left(\xi\rho^{J} - \frac{1}{\xi\rho^{J}}\right)^{p} = \sum_{q=0}^{p} \binom{p}{q} \xi^{2q-p} (-1)^{p-q} \rho^{J(2q-p-m)} \tag{A.4}$$

which we insert into (A.2). After summing over I, J, and rearranging the terms, we obtain

$$Z_{\ell m}(X) = 2^{-k} \zeta^{\ell} \xi^{m} k! \sum_{q}^{k} \frac{(-1)^{q-m} (\cos\chi)^{k-2q+m} (\sin\chi)^{2q-m}}{q!(q-m)!\left(\frac{k+\ell-2q+m}{2}\right)!\left(\frac{k-\ell-2q+m}{2}\right)!}. \tag{A.5}$$

This formula results from the fact that, through (9), the summations over I, J imply $p = 2q - m$ and $2r = \ell + k + m - 2q$, which we have reported. The range of the summation over q is defined by the conditions

$$0 \leqslant \ell + k + m - 2q \leqslant 2k + 2m - 4q \leqslant 2k \qquad 0 \leqslant q \leqslant 2q - m \leqslant k. \tag{A.6}$$

Rearrangements of the previous formula, inserting $u \equiv \cos(2\chi) = 2\cos^2\chi - 1 = 1 - 2\sin^2\chi$, lead to

$$Z_{\ell m}(X) = \frac{2^{-3k/2} \zeta^{\ell} \xi^{m} k! (1+u)^{\frac{\ell}{2}} (1-u)^{\frac{m}{2}}}{(m+d)!(\ell+d)!}$$
$$\sum_{q} \binom{m+d}{i} \binom{\ell+d}{d-i} (1+u)^{i} (u-1)^{d-i} \tag{A.7}$$

where we have defined $i \equiv \frac{k+m-\ell}{2} - q$ and $d \equiv \frac{k-\ell-m}{2}$. Verification shows that the range defined above gives exactly the expansion formula for the Jacobi polynomial. The comparison with (11) gives the coefficient

$$P_{k;m_1,m_2} = \frac{2^{-k} k!}{(k/2 - m_1)!(k/2 + m_1)!(k+1)^2 C_{k;m_1,m_2}}$$
$$= \frac{2^{-k} \pi k! (k+1)^{-5/2}}{\sqrt{(k/2+m_2)!(k/2-m_2)!(k/2+m_1)!(k/2-m_1)!}}.$$

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